

GARDNER'S DEFORMATIONS OF THE GRADED KORTEWEG-DE VRIES EQUATIONS REVISITED

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ABSTRACT. We solve the Gardner deformation problem for the $N=2$ supersymmetric $a=4$ Korteweg-de Vries equation (P. Mathieu, 1988). We show that a known zero-curvature representation for this superequation yields the system of new nonlocal variables such that their derivatives contain the Gardner deformation for the classical KdV equation.

Introduction. The classical problem of construction of the Gardner deformation [1] for an infinite-dimensional completely integrable system of evolutionary partial differential equations essentially amounts to finding a recurrence relation between the integrals of motion. For the $N=2$ supersymmetric generalizations of the Korteweg-de Vries equation [2, 3], the deformation problem was posed when the integrable triplet of such super-systems was discovered. Various attempts to solve it were undertaken since then (e.g., see [2]) but the progress was limited. The first solution for the $N=2$, $a=4$ SKdV in the triplet $a \in \{-2, 1, 4\}$ was achieved in [4]; in that paper, we stated the ‘no-go’ theorem about the impossibility to deform this super-equation in terms of the superfield (but not the impossibility to deform it by treating the components of the super-field separately, see below), c.f. [2]. We then presented the two-step solution of the deformation problem: We obtained the Gardner deformation for the Kaup-Boussinesq equation, which is the bosonic limit of the super-equation that precedes the $N=2$, $a=4$ super-KdV in its hierarchy. We thus derived the recurrence relation between the Hamiltonians of the bosonic limit hierarchy and then we showed how each conserved density is extended to the super-density for the $N=2$ super-system. In other words, we deformed the bosonic subsystem of the super-equation at hand within the frames of the classical scheme [1], whence we recovered the full $N=2$ supersymmetry-invariance.

In this paper we re-address, from a principally different viewpoint, the Gardner deformation problem for a vast class of (not necessarily supersymmetric) KdV-like systems. Namely, in [4] we emphasized the geometric likeness of the Gardner deformations and zero-curvature representations, each of them manifesting the integrability of nonlinear systems. Indeed, both constructions generate infinite sequences of nontrivial integrals

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of motion. We stress that, in general, the two constructions are *not* equivalent, although they provide similar results. It is precisely this correspondence which we study in this paper for the graded KdV systems.

Developing further the approach of [5], we reformulate the Gardner deformation problem for the graded extensions of the KdV equation in terms of constructing parameter-dependent families of new bosonic and fermionic variables. We require that the ‘nonlocalities’ possess two defining properties ([6, 7]): on one hand, they should reproduce the classical Gardner deformation from [1] under the shrinking of the $N=2$ super-equation back to the KdV equation. On the other hand, we consider the nonlocalities that encode the parameter-dependent zero-curvature representations for the super-systems at hand. In this reformulation, we solve P. Mathieu’s Open problem 2 of [3] for the $N=2$ supersymmetric $a=4$ -KdV equation. However, our approach is applicable to a much wider class of completely integrable (super-)systems.

This paper is structured as follows. We first fix some notation and compare the notions of Gardner’s deformations and zero-curvature representations by using their unifying description in terms of nonlocal structures for partial differential equations (PDE). In section 2 we proceed with this correspondence for \mathbb{Z}_2 -graded systems of evolutionary PDE and solve the Gardner deformation problem for the $N=2$, $a=4$ SKdV (29). The nature of the new solution is geometric and it presents an alternative to the analytic two-step algorithm that works for graded systems and which we reported earlier in [4].

1. PRELIMINARIES

In this section we fix some notation, which follows [8, 9, 10]. Let Σ^n be an n -dimensional manifold, $1 \leq n < \infty$, let $\pi: E^{m+n} \rightarrow \Sigma^n$ be a vector bundle over Σ^n of fiber dimension m . Consider the jet space $J^\infty(\pi)$ of sections of the vector bundle π . The local coordinates on $J^\infty(\pi)$ are composed by the coordinates x^i on Σ^n , coordinates u^j along the fibers of π , and coordinates u_σ^j along the fibers of $J^\infty(\pi) \rightarrow \Sigma$; here σ is a multi-index and $u_\sigma^j \equiv u^j$. The commuting vector fields $\frac{d|\sigma|}{dx^\sigma} = \partial/\partial x_i + \sum_{k,\sigma} u_{\sigma i}^k \partial/\partial u_\sigma^k$ on $J^\infty(\pi)$ are called *total derivatives*. The operators $\frac{d|\sigma|}{dx^\sigma}$ that act on the space of smooth functions on $J^\infty(\pi)$ are defined inductively by the formula $\frac{d|\sigma i|}{dx^{\sigma i}} = \frac{d|\sigma|}{dx^\sigma} \circ \frac{d}{dx^i}$.

Consider a system \mathcal{E} of r partial differential equations,

$$\mathcal{E} = \{F^\ell(x_i, u, \dots, u_i^k, \dots) = 0, \quad \ell = 1, \dots, r\}.$$

The system $\{F^\ell = 0\}$ and all its differential consequences $\frac{d|\sigma|}{dx^\sigma} F^\ell = 0$, $|\sigma| \geq 1$ generate the infinite prolongation of \mathcal{E} , which we denote by \mathcal{E}^∞ . The restrictions of $\frac{d}{dx^i}$ on \mathcal{E}^∞ determine the Cartan distribution \mathcal{C} on the tangent space $T\mathcal{E}^\infty$. Here and in what follows, the notation $\frac{d^x}{dx^x}$ stands for the restrictions of the total derivatives onto \mathcal{E}^∞ . There is the decomposition of the tangent space $T\mathcal{E}^\infty$ to the direct sum of the horizontal (the Cartan distribution) and the vertical vector spaces, $T\mathcal{E}^\infty = \mathcal{C} \oplus V\mathcal{E}^\infty$. Let $\Lambda^{1,0}(\mathcal{E}^\infty) = \text{Ann } \mathcal{C}$ and $\Lambda^{0,1}(\mathcal{E}^\infty) = \text{Ann } V\mathcal{E}^\infty$ be the $C^\infty(\mathcal{E}^\infty)$ -modules of contact 1-forms and horizontal 1-forms vanishing on \mathcal{C} and $V\mathcal{E}^\infty$, respectively. Let $\Lambda^r(\mathcal{E}^\infty)$ denote the C^∞ -module of r -forms on \mathcal{E}^∞ . We have the decomposition $\Lambda^r(\mathcal{E}^\infty) = \bigoplus_{q+p=r} \Lambda^{p,q}(\mathcal{E}^\infty)$, where $\Lambda^{p,q}(\mathcal{E}^\infty) = \bigwedge^p \Lambda^{1,0}(\mathcal{E}^\infty) \wedge \bigwedge^q \Lambda^{0,1}(\mathcal{E}^\infty)$. According to this decomposition, the exterior differential splits to the sum $d = \bar{d} + d_{\mathcal{C}}$

of the horizontal differential $\bar{d} : \Lambda^{p,q}(\mathcal{E}^\infty) \rightarrow \Lambda^{p,q+1}(\mathcal{E}^\infty)$ and the vertical differential $d_{\mathcal{C}} : \Lambda^{p,q}(\mathcal{E}^\infty) \rightarrow \Lambda^{p+1,q}(\mathcal{E}^\infty)$.

The differentials $d_{\mathcal{C}}$ and \bar{d} can be expressed in coordinates by their actions on the elements $\phi \in C^\infty(\mathcal{E}^\infty) = \Lambda^{0,0}(\mathcal{E}^\infty)$, whence

$$\bar{d}\phi = \sum_i \frac{d}{dx^i} \phi dx^i, \quad d_{\mathcal{C}}\phi = \sum_{\sigma,k} \frac{\partial \phi}{\partial u_\sigma^k} \omega_\sigma^k,$$

here $\omega_\sigma^k = du_\sigma^k - \sum_i u_{\sigma i}^k dx^i$ are the Cartan forms.

1.1. Zero curvature representations and coverings. Consider the tensor product over \mathbb{C} of the exterior algebra $\bar{\Lambda}(\mathcal{E}^\infty) = \bigwedge^* \Lambda^{i,0}(\mathcal{E}^\infty)$ with a finite-dimensional complex Lie algebra \mathfrak{g} . The product is endowed with the bracket $[A\mu, B\nu] = [A, B]\mu \wedge \nu$ for $\mu, \nu \in \bar{\Lambda}(\mathcal{E}^\infty)$ and $A, B \in \mathfrak{g}$. The tensor product $\bar{\Lambda}(\mathcal{E}^\infty) \otimes \mathfrak{g}(\mathbb{C})$ is a differential graded associative algebra with respect to the multiplication $A\mu \cdot B\nu = (A \cdot B)\mu \wedge \nu$ induced by the ordinary matrix multiplication so that

$$\bar{d}(\rho \cdot \sigma) = \bar{d}\rho \cdot \sigma + (-1)^r \rho \cdot \bar{d}\sigma$$

for $\rho \in \bar{\Lambda}^r(\mathcal{E}^\infty) \otimes \mathfrak{g}(\mathbb{C})$ and $\sigma \in \bar{\Lambda}^s(\mathcal{E}^\infty) \otimes \mathfrak{g}(\mathbb{C})$, whereas

$$[\rho, \sigma] = \rho \cdot \sigma - (-1)^{rs} \sigma \cdot \rho.$$

The elements of $C^\infty(\mathcal{E}^\infty) \otimes \mathfrak{g}$ will be called the **\mathfrak{g} -matrices** [8].

Definition 1 ([8]). A horizontal 1-form $\alpha \in \bar{\Lambda}^1(\mathcal{E}^\infty) \otimes \mathfrak{g}$ is called the **\mathfrak{g} -valued zero-curvature representation** for the equation \mathcal{E} if the Maurer–Cartan condition holds:

$$\bar{d}\alpha = \frac{1}{2}[\alpha, \alpha]. \quad (1)$$

If $\alpha = \sum_{i=1}^n A_i dx^i$, where $A_i \in \mathfrak{g}$, is a \mathfrak{g} -valued zero-curvature representation for \mathcal{E} , then we have

$$0 = \bar{d}\alpha - \frac{1}{2}[\alpha, \alpha] = \sum_{i < j} \left(\frac{d}{dx^i} A_j - \frac{d}{dx^j} A_i \right) dx^i \wedge dx^j - \sum_{i < j} [A_i, A_j] dx^i \wedge dx^j = 0.$$

Therefore, equation (1) is equivalent to the following set of conditions upon the \mathfrak{g} -matrices A_i :

$$\frac{d}{dx^j} A_i - \frac{d}{dx^i} A_j + [A_i, A_j] = 0, \quad \forall i, j = 1, \dots, m : i \neq j. \quad (2)$$

The most interesting zero-curvature representations for \mathcal{E} are those which contain a non-removable spectral parameter; in this case the system \mathcal{E} is integrable. (The parameter is *removable* if one obtains gauge-equivalent zero-curvature representations, see section 1.3 for details, at different values of the parameter; otherwise, the parameter is non-removable, or “essential.”)

We recall that n is the dimension of the base Σ^n for the vector bundle π . From now on, we consider mainly $(1+1)$ -dimensional systems, *i.e.*, we let $n = 2$ and interpret one independent variable as the time t and the other as the spatial coordinate x . With the conventions $n = 2$, $x^1 = x$, $x^2 = t$, $A_1 = A$, and $A_2 = B$, the Maurer–Cartan equations (1–2) read

$$\frac{d}{dt} A - \frac{d}{dx} B + [A, B] = 0. \quad (2')$$

This is the compatibility condition for the auxiliary linear system

$$\Psi_x = A\Psi, \quad \Psi_t = B\Psi,$$

where Ψ is a wave function and matrices A and B belong to the tensor product of a matrix Lie algebra \mathfrak{g} and the ring $C^\infty(\mathcal{E}^\infty)$ of smooth functions on the prolongation \mathcal{E}^∞ . If the matrices A and B depend on the spectral parameter and it is non-removable, then the equation \mathcal{E} is integrable [11].

The construction of Gardner's deformations [1] is another way to prove the integrability of evolution equations \mathcal{E} .

Definition 2 (Gardner's deformation [1]). Let $\mathcal{E} = \{u_t = f(x, u, u_x, \dots, u_k)\}$ be a system of evolutionary partial differential equations upon the unknowns $u(x, t)$ in two variables. Let $\mathcal{E}_\varepsilon = \{\tilde{u}_t = f_\varepsilon(x, \tilde{u}, \tilde{u}_x, \dots, \tilde{u}_k; \varepsilon) \mid f_\varepsilon \in \text{im } \frac{d}{dx}\}$ be a deformation of \mathcal{E} such that at each point $\varepsilon \in \mathcal{I} \subseteq \mathbb{R}$ there exists the Miura contraction $\mathbf{m}_\varepsilon = \{u = u(\tilde{u}, \tilde{u}_x, \dots; \varepsilon) : \mathcal{E}_\varepsilon \rightarrow \mathcal{E}\}$. Then the pair $(\mathcal{E}_\varepsilon, \mathbf{m}_\varepsilon)$ is the *Gardner deformation* for the system \mathcal{E} .

One obtains the recurrence relations between the conserved densities $\tilde{u}_n(x, u, u_x, \dots)$ for \mathcal{E} using the contraction \mathbf{m}_ε and the expansion $\tilde{u} = \sum_{n=0}^{+\infty} \tilde{u}_n \varepsilon^n$ of the deformed unknowns \tilde{u} in ε .

In the recent paper [7] we understood Gardner's deformations in the extended sense, namely, in terms of coverings over PDE and diagrams of coverings. The zero-curvature representations and Gardner's deformations can be considered as such geometric structures¹ that obey some extra conditions.

Definition 3 ([10]). Let the assumptions on p. 2 hold and let the notation be as previously fixed. A *covering* over the equation \mathcal{E} is another (usually, larger) system of partial differential equations $\tilde{\mathcal{E}}$ endowed with the n -dimensional Cartan distribution $\tilde{\mathcal{C}}$ and such that there is a mapping $\tau : \tilde{\mathcal{E}} \rightarrow \mathcal{E}^\infty$ for which, at each point $\theta \in \tilde{\mathcal{E}}$ the tangent map $\tau_{*,\theta}$ is an isomorphism of the plane $\tilde{\mathcal{C}}_\theta$ to the Cartan plane $\mathcal{C}_{\tau(\theta)}$ at the point $\tau(\theta)$ in \mathcal{E}^∞ .

In practice, the construction of a covering over \mathcal{E} means the introduction of new non-local variables such that their compatibility conditions lie inside the initial system \mathcal{E}^∞ . Whenever the covering is indeed realized as the fibre bundle $\tau : \tilde{\mathcal{E}} \rightarrow \mathcal{E}$, the forgetful map τ discards the nonlocalities.

In these terms, the zero-curvature representations and Gardner's deformations are coverings of special kinds (see Examples 2 and 4 below). In this paper we use the geometric closedness of the two notions and construct new Gardner's deformations from known zero-curvature representations (but this is *not always* possible).

Example 1 (A zero-curvature representation for the KdV equation). Consider the Korteweg–de Vries (KdV) equation [1]

$$\mathcal{E}_{\text{KdV}} = \{u_t + u_{xxx} + 6uu_x = 0\} \quad (3)$$

¹Bäcklund (auto)transformations between PDE appear in the same context. In [7] we argued that the former, when regarded as the diagrams, are dual to the diagram description of Gardner's deformations.

and its Lax representation [1, 9, 11]

$$\mathcal{L}_t = [\mathcal{L}, \mathcal{A}],$$

where

$$\mathcal{L} = \frac{d^2}{dx^2} + u, \quad \mathcal{A} = -4\frac{d^3}{dx^3} - 6u\frac{d}{dx} - 3u_x. \quad (4)$$

The linear auxiliary problem [12] is

$$\begin{aligned} \psi_{xx} + u\psi &= \lambda\psi, \\ -4\psi_{xxx} - 6u\psi_x - 3u_x\psi &= \psi_t, \end{aligned}$$

By definition, put $\psi_0 = \psi$ and $\psi_1 = \psi_x$. We obtain

$$\begin{aligned} \psi_{0;x} &= \psi_1, \\ \psi_{1;x} &= (\lambda - u)\psi_0, \\ \psi_{0;t} &= -4\frac{d}{dx}((\lambda - u)\psi_0) - 6u\psi_1 - 3u_x\psi_0 = u_x\psi_0 + (-4\lambda - 2u)\psi_1, \\ \psi_{1;t} &= (-4\lambda^2 + 2u\lambda + 2u^2 + u_x)\psi_0 + (-u_x)\psi_1. \end{aligned}$$

We finally rewrite this system as two matrix equations [12],

$$\underbrace{\begin{pmatrix} \psi_{0;x} \\ \psi_{1;x} \end{pmatrix}}_{\psi_x} = \underbrace{\begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix}}_A \underbrace{\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}}_{\psi} \quad \underbrace{\begin{pmatrix} \psi_{0;t} \\ \psi_{1;t} \end{pmatrix}}_{\psi_t} = \underbrace{\begin{pmatrix} u_x & -4\lambda - 2u \\ -4\lambda^2 + 2u\lambda + 2u^2 + u_x & -u_x \end{pmatrix}}_B \underbrace{\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}}_{\psi}$$

This yields an $\mathfrak{sl}_2(\mathbb{C})$ -valued zero-curvature representation $\alpha^{\text{KdV}} = A dx + B dt$ for the KdV equation (3). The representation α^{KdV} was rediscovered in [13].

Example 2 (Zero-curvature representations as coverings). Let $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ as above. We introduce the standard basis e, h, f in \mathfrak{g} such that

$$[e, h] = -2e, \quad [e, f] = h, \quad [f, h] = 2f.$$

We consider, simultaneously, the matrix representation

$$\rho : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \{M \in \text{Mat}(2, 2) \mid \text{tr } M = 0\}$$

of \mathfrak{g} and its representation in the space of vector fields with polynomial coefficients on the complex line with the coordinate w :

$$\begin{aligned} \rho(e) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & \rho(h) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \rho(f) &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \varrho(e) &= 1 \cdot \partial / \partial w, & \varrho(h) &= -2w \cdot \partial / \partial w, & \varrho(f) &= -w^2 \cdot \partial / \partial w. \end{aligned}$$

Let us decompose the matrices A_i (which occur in the zero-curvature representation $\alpha = \sum_i A_i dx^i$) with respect to the basis in the space $\rho(\mathfrak{g})$,

$$A_i = a_e^{(i)} \otimes \rho(e) + a_h^{(i)} \otimes \rho(h) + a_f^{(i)} \otimes \rho(f), \quad (5)$$

for $a_j^{(i)} \in C^\infty(\mathcal{E}^\infty)$.

To construct the covering $\tilde{\mathcal{E}}$ over \mathcal{E}^∞ with a new nonlocal variable w , we switch from the representation ρ to ϱ . We thus obtain the vector fields

$$V_{A_i} = a_e^{(i)} \otimes \varrho(e) + a_h^{(i)} \otimes \varrho(h) + a_f^{(i)} \otimes \varrho(f) \quad (5')$$

such that the prolongations of the total derivatives $\frac{d^i}{dx^i}$ to $\tilde{\mathcal{E}}$ are defined by the formula

$$\frac{\tilde{d}}{dx^i} = \frac{d}{dx^i} - V_{A_i}. \quad (6)$$

The extended derivatives act on the nonlocal variable w as follows

$$\frac{\tilde{d}}{dx^i} w = dw \lrcorner (-V_{A_i}).$$

Remark 1. The commutativity of the prolonged total derivatives $[\frac{\tilde{d}}{dx^i}, \frac{\tilde{d}}{dx^j}] = 0$ with $i \neq j$ is equivalent to the Maurer–Cartan equation (2): Indeed, we have that

$$\begin{aligned} 0 &= [\frac{\tilde{d}}{dx^i}, \frac{\tilde{d}}{dx^j}] = [\frac{d}{dx^i} - V_{A_i}, \frac{d}{dx^j} - V_{A_j}] = [\frac{d}{dx^i}, \frac{d}{dx^j}] - [\frac{d}{dx^i}, V_{A_j}] - [V_{A_j}, \frac{d}{dx^j}] + [V_{A_i}, V_{A_j}] = \\ &= -V_{\frac{d}{dx^i} A_j} + V_{\frac{d}{dx^j} A_i} + V_{[A_i, A_j]} = V_{\frac{d}{dx^j} A_i - \frac{d}{dx^i} A_j + [A_i, A_j]} \Leftrightarrow \frac{d}{dx^j} A_i - \frac{d}{dx^i} A_j + [A_i, A_j] = 0. \end{aligned}$$

This motivates the choice of the minus sign in (6).

Example 3 (A one-dimensional covering over the KdV equation). One obtains the covering over the KdV equation from the zero-curvature representation α (see Example 1) by using representation (5') in the space of vector fields. Applying (5') to the matrices $A, B \in \mathfrak{sl}_2(\mathbb{C})$, we construct the following vector fields with the nonlocal variable w :

$$V_A = (1 - (\lambda - u)w^2) \cdot \partial / \partial w,$$

$$V_B = [(-4\lambda - 2u) - 2uw - (-4\lambda^2 + 2u\lambda + 2u^2 + u_{xx})w^2] \cdot \partial / \partial w.$$

The prolongations of the total derivatives act on w by the rules

$$w_x = -1 + (\lambda - u)w^2, \quad (7a)$$

$$w_t = -((-4\lambda - 2u) - 2u_x w - (-4\lambda^2 + 2u\lambda + 2u^2 + u_{xx})w^2). \quad (7b)$$

We thus obtain the one-dimensional covering over the KdV equation (3). It depends on the non-removable [10] spectral parameter λ . In what follows we show that this covering is equivalent to the covering (12) which is derived from Gardner's deformation (11) of the KdV equation (3).

1.2. The projective substitution and nonlinear representations of Lie algebras in the spaces of vector fields. Suppose \mathfrak{g} is a finite-dimensional Lie algebra. We shall use the projective substitution [5] to construct a covering over the equation \mathcal{E} starting from a \mathfrak{g} -valued zero-curvature representation for \mathcal{E} .

Let M be an m -dimensional manifold with local coordinates

$$v = (v^1, v^1, \dots, v^m) \in M, \text{ and put } \partial_v = (\partial_{v^1}, \partial_{v^2}, \dots, \partial_{v^m})^t.$$

For any $g \in \mathfrak{gl} \subseteq \mathfrak{gl}_n(\mathbb{C})$, its representation V_g in the space of vector fields on M is given by the formula

$$V_g = v^t g \partial_v.$$

We note that V_g is linear in v^i . By construction, the representation preserves the commutation relations in the initial Lie algebra \mathfrak{g} :

$$[V_g, V_f] = [v^t g \partial_v, v^t f \partial_v] = v^t [g, f] \partial_v = V_{[g, f]}, \quad f, g \in \mathfrak{g}.$$

At all points of M where $v_1 \neq 0$ we consider the projection

$$\pi : v^i \rightarrow w^i = \mu v^i / v^1, \quad \mu \in \mathbb{R} \quad (8)$$

and its differential $d\pi : \partial_v \rightarrow \partial_w$. The transformation π yields the new coordinates on the open subset of M where $v^1 \neq 0$ and corresponding subset of TM :

$$w = (\mu, w^2, \dots, w^m), \quad \partial_w = \left(-\frac{1}{\mu} \sum_{i=2}^m w^i \partial_{w^i}, \partial_{w^2}, \dots, \partial_{w^m}\right).$$

Consider the vector field $W_g = d\pi(V_g)$. In coordinates, we have

$$W_g = wg\partial_w^t. \quad (9)$$

We note that, generally, W_g is nonlinear with respect to w^i . The commutation relations between the vector fields of such type are also inherited from the relations in the Lie algebra \mathfrak{g} :

$$[W_g, W_f] = [d\pi(V_g), d\pi(V_f)] = d\pi([g, f]) = d\pi(V_{[g, f]}) = W_{[g, f]}.$$

Using representation (9) for the matrices A and B that determine the zero-curvature representation $\alpha^{\text{KdV}} = Adx + Bdt$ for the KdV equation, we obtain their realizations in terms of the vector fields:

$$W_A = \frac{1}{\mu}(-\lambda w^2 + \mu^2 + uw^2) \partial / \partial w,$$

$$W_B = \frac{1}{\mu}(-u_{xx}w^2 - 2u_x\mu w + 4\lambda^2 w^2 - 4\lambda\mu^2 - 2\lambda uw^2 - 2\mu^2 u - 2u^2 w^2) \partial / \partial w.$$

Therefore, the prolongations of the total derivatives act on the nonlocality w as follows:

$$w_x = -\frac{1}{\mu}(-\lambda w^2 + \mu^2 + uw^2), \quad (10a)$$

$$w_t = -\frac{1}{\mu}(-u_{xx}w^2 - 2u_x\mu w + 4\lambda^2 w^2 - 4\lambda\mu^2 - 2\lambda uw^2 - 2\mu^2 u - 2u^2 w^2). \quad (10b)$$

The parameter μ is removable by the transformation $w \rightarrow \mu w$, which rescales it to unit. Applying this transformation to (10), we reproduce the covering (7).

Example 4 (A covering which is based on Gardner's deformation). Consider the Gardner deformation [1] of the KdV equation (3),

$$\mathcal{E}_\varepsilon = \{\tilde{u}_t = -(\tilde{u}_{xx} + 3\tilde{u}^2 - 2\varepsilon^2 \tilde{u}^3)_x\}, \quad (11a)$$

$$\mathfrak{m}_\varepsilon = \{u = \tilde{u} - \varepsilon \tilde{u}_x - \varepsilon^2 \tilde{u}^2\} : \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_0. \quad (11b)$$

Expressing \tilde{u}_x from (11b) and substituting it in (11a), we obtain the one-dimensional covering over the KdV equation,

$$\tilde{u}_x = \frac{1}{\varepsilon}(\tilde{u} - u) - \varepsilon \tilde{u}^2, \quad (12a)$$

$$\tilde{u}_t = \frac{1}{\varepsilon}(u_{xx} + 2u^2) + \frac{1}{\varepsilon^2}u_x + \frac{1}{\varepsilon^3}u + \left(-2u_x - \frac{2}{\varepsilon}u - \frac{1}{\varepsilon^3}\right)\tilde{u} + \left(2\varepsilon u + \frac{1}{\varepsilon}\right)\tilde{u}^2, \quad (12b)$$

We claim that covering (12) is equivalent to the covering that was obtained in [10, p. 277] for the KdV equation. To prove this, we first put $\tilde{u} = -\tilde{v}/\varepsilon$. We have

$$-\frac{\tilde{v}}{\varepsilon} = -\frac{1}{\varepsilon^2}\tilde{v} - \frac{1}{\varepsilon}u - \frac{1}{\varepsilon}\tilde{v}^2,$$

in other words

$$\tilde{v}_x = u + \left(\tilde{v} + \frac{1}{2\varepsilon} \right)^2 - \frac{1}{4\varepsilon^2}.$$

Next, we put $p = \tilde{v} + 1/(2\varepsilon)$, whence we obtain

$$p_x = u + p^2 - \frac{1}{4\varepsilon^2}, \quad (13a)$$

$$p_t = -u_{xx} - 2u^2 - \frac{1}{2\varepsilon^2}u + \frac{1}{4\varepsilon^4} - 2u_x p - (\varepsilon^2 u + \frac{1}{2})p^2. \quad (13b)$$

Dividing (7) by w^2 , we conclude that

$$\begin{aligned} w_x &= -1 + (\lambda - u)w^2, \\ \frac{w_x}{w^2} &= -\frac{1}{w^2} - u + \lambda. \end{aligned}$$

On the other hand, we put $p = 1/w$, whence $p_x = -w_x/w^2$, and set $\lambda = 1/(4\varepsilon^2)$. This brings (7) to the same notation as in formulas (13),

$$\begin{aligned} p_x &= u + p^2 - \frac{1}{4\varepsilon^2}, \\ p_x &= u + p^2 - \lambda. \end{aligned}$$

The corresponding one-form of the zero-curvature representation for the KdV equation is equal to

$$\alpha_2^{\text{KdV}} = \begin{pmatrix} 0 & \lambda - u \\ 1 & 0 \end{pmatrix} dx + \begin{pmatrix} -u_x & -4\lambda^2 + 2\lambda u + 2u^2 + u_{xx} \\ -4\lambda - 2u & u_x \end{pmatrix} dt. \quad (14)$$

In the next section we show that this zero curvature representation is also equivalent to α^{KdV} from Example 1.

1.3. Gauge transformations. Let G be the Lie group of the Lie algebra \mathfrak{g} (so that $G = SL_2(\mathbb{C})$ in the previous example). Given an equation \mathcal{E} , for any-zero curvature representation α there exists the zero-curvature representation α^S such that

$$\alpha^S = \bar{d}S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1}, \quad S \in C^\infty(\mathcal{E}^\infty) \otimes G. \quad (15)$$

The zero-curvature representation α^S is called *gauge-equivalent* to α and S is the *gauge transformation*. Suppose $\alpha = A_i dx^i$. The gauge transformation S acts on the components A_i of α as follows

$$A_i^S = \frac{d}{dx^i}(S)S^{-1} + SA_iS^{-1}. \quad (15')$$

Example 5 (The relation between the coverings which stem from gauge equivalent zero curvature representations). Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and $G = SL_2(\mathbb{C})$. Suppose $S \in SL_2(\mathbb{C})$, so that

$$S = \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}, \quad \det S = 1.$$

Let $\alpha = \sum_i A_i dx^i$ be a zero-curvature representation for an equation \mathcal{E} . Using decomposition (5) for $A_i \in \mathfrak{sl}_2(\mathbb{C})$, we inspect how the gauge transformation S acts on the

components of α :

$$\begin{aligned}
A_i^S &= \frac{d}{dx^i}(S)S^{-1} + S(a_e^{(i)} \otimes \rho(e) + a_h^{(i)} \otimes \rho(h) + a_f^{(i)} \otimes \rho(f))S^{-1} = \\
&= \frac{d}{dx^i}(S)S^{-1} + a_e^{(i)} \otimes (S \cdot \rho(e) \cdot S^{-1}) + a_h^{(i)} \otimes (S \cdot \rho(h) \cdot S^{-1}) + a_f^{(i)} \otimes (S \cdot \rho(f) \cdot S^{-1}), \\
\frac{d}{dx^i}(S)S^{-1} &= \begin{pmatrix} s_{1;i}s_4 - s_{2;i}s_3 & s_{2;i}s_1 - s_{1;i}s_2 \\ s_{3;i}s_4 - s_{4;i}s_3 & s_{4;i}s_1 - s_{3;i}s_2 \end{pmatrix} = \begin{pmatrix} s_{1;i}s_4 - s_{2;i}s_3 & s_{2;i}s_1 - s_{1;i}s_2 \\ s_{3;i}s_4 - s_{4;i}s_3 & -s_{1;i}s_4 + s_{2;i}s_3 \end{pmatrix} = \\
&= (s_{2;i}s_1 - s_{1;i}s_2)\rho(e) + (s_{1;i}s_4 - s_{2;i}s_3)\rho(h) + (s_{3;i}s_4 - s_{4;i}s_3)\rho(f) \\
S \cdot \rho(e) \cdot S^{-1} &= \begin{pmatrix} -s_1s_3 & s_1^2 \\ -s_3^2 & s_1s_3 \end{pmatrix} = (s_1^2)\rho(e) + (-s_1s_3)\rho(h) + (-s_3^2)\rho(f), \\
S \cdot \rho(h) \cdot S^{-1} &= \begin{pmatrix} s_1s_4 + s_2s_3 & -2s_1s_2 \\ 2s_3s_4 & -s_1s_4 - s_2s_3 \end{pmatrix} = (-2s_1s_2)\rho(e) + (s_1s_4 + s_2s_3)\rho(h) + (2s_3s_4)\rho(f), \\
S \cdot \rho(f) \cdot S^{-1} &= \begin{pmatrix} s_2s_4 & -s_2^2 \\ s_4^2 & -s_2s_4 \end{pmatrix} = (-s_2^2)\rho(e) + (s_2s_4)\rho(h) + (s_4^2)\rho(f),
\end{aligned}$$

We finally obtain

$$\begin{aligned}
A_i^S &= (s_{2;i}s_1 - s_{1;i}s_2 + s_1^2a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2a_f^{(i)}) \otimes \rho(e) + \\
&+ (s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)}) \otimes \rho(h) + \\
&+ (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)}) \otimes \rho(f).
\end{aligned}$$

Passing to the vector field representation of A_i^S by using formula (5'), we have

$$\begin{aligned}
V_{A_i^S} &= (s_{2;i}s_1 - s_{1;i}s_2 + s_1^2a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2a_f^{(i)}) \otimes \varrho(e) + \\
&+ (s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)}) \otimes \varrho(h) + \\
&+ (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)}) \otimes \varrho(f). \quad (16)
\end{aligned}$$

In other words, whenever we start from the covering of \mathcal{E} associated with a zero-curvature representation α , such that the differentiation rules for the nonlocality w are

$$\frac{d}{dx^i}(w) = -a_e^{(i)} + 2a_h^{(i)}w + a_f^{(i)}w^2,$$

we obtain the covering which is associated with α^S :

$$\begin{aligned}
\frac{d}{dx^i}(w_S) &= -(s_{2;i}s_1 - s_{1;i}s_2 + s_1^2a_e^{(i)} - 2s_1s_2a_h^{(i)} - s_2^2a_f^{(i)}) + \\
&+ 2(s_{1;i}s_4 - s_{2;i}s_3 - s_1s_3a_e^{(i)} + (s_1s_4 + s_2s_3)a_h^{(i)} + s_2s_4a_f^{(i)})w_S + \\
&+ (s_{3;i}s_4 - s_{4;i}s_3 - s_3^2a_e^{(i)} + 2s_3s_4a_h^{(i)} + s_4^2a_f^{(i)})w_S^2. \quad (17)
\end{aligned}$$

We shall use this relations between the two coverings in the search of the gauge transformations between known zero-curvature representations for the KdV equation.

Example 6 (Gauge transformations between zero-curvature representations for the KdV equation). Let us find the gauge transformations that bring coverings (7) and (12) to the form (13).

For the transformation (7)→(13) we have

$$\begin{aligned} p_x = u + p^2 - \lambda = & -(s_{2;x}s_1 - s_{1;x}s_2 + s_1^2 - s_2^2(\lambda - u) + \\ & - 2(s_{1;x}s_4 - s_{2;x}s_3 - s_1s_3 + s_2s_4(\lambda - u))p - \\ & - (s_{3;x}s_4 - s_{4;x}s_3 - s_3^2 + s_4^2(\lambda - u))p^2). \end{aligned}$$

Solving this equation for s_i , we find a unique solution $s_2 = s_3 = i$, $s_1 = s_4 = 0$:

$$S = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}. \quad (18)$$

The matrices of the zero curvature representations (7) and (13) are related as follows:

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda - u & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \lambda - u \\ 1 & 0 \end{pmatrix}.$$

On the other hand, for the transformation (12)→(13) we have

$$\begin{aligned} p_x = u + p^2 - \frac{1}{4\varepsilon^2} = & -(s_{2;x}s_1 - s_{1;x}s_2 - s_1^2\frac{u}{\varepsilon} + s_1s_2\frac{1}{\varepsilon} - s_2^2\varepsilon + \\ & - 2(s_{1;x}s_4 - s_{2;x}s_3 + s_1s_3\frac{u}{\varepsilon} - (s_1s_4 + s_2s_3)\frac{1}{2\varepsilon} + s_2s_4\varepsilon)p - \\ & - (s_{3;x}s_4 - s_{4;x}s_3 + s_3^2\frac{u}{\varepsilon} - s_3s_4\frac{1}{\varepsilon} + s_4^2\varepsilon)p^2). \end{aligned}$$

Solving this equation for s_i , we find a solution $s_1 = i/\sqrt{\varepsilon}$, $s_2 = i/(2\varepsilon\sqrt{\varepsilon})$, $s_3 = 0$, $s_4 = i\sqrt{\varepsilon}$. Therefore,

$$S = \begin{pmatrix} i/\sqrt{\varepsilon} & i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & -i\sqrt{\varepsilon} \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} -i\sqrt{\varepsilon} & -i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & i/\sqrt{\varepsilon} \end{pmatrix}, \quad (19)$$

The matrices of the zero-curvature representations (12) and (13) satisfy the relation

$$\begin{pmatrix} i/\sqrt{\varepsilon} & i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & -i\sqrt{\varepsilon} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{4\varepsilon^2} - u \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -i\sqrt{\varepsilon} & -i/(2\varepsilon\sqrt{\varepsilon}) \\ 0 & i/\sqrt{\varepsilon} \end{pmatrix} = \begin{pmatrix} \frac{1}{2\varepsilon} & \frac{u}{\varepsilon} \\ -\varepsilon & -\frac{1}{2\varepsilon} \end{pmatrix}.$$

Let us recall that in Example 1 we derived the zero-curvature representation for the KdV equation from its Lax pair. Having done that, we also revised the transition from this zero-curvature representation to the Gardner deformation of the KdV equation. In the next section we extend this approach and find the generalizations of Gardner's deformation (11) for graded systems, in particular, for the $N=1$ and $N=2$ supersymmetric Korteweg–de Vries equations.

2. GRADED SYSTEMS

2.1. Lie super-algebras. We recall first the definition of the Lie super-algebra [14, 15, 16]. Let \mathcal{A} be an algebra over the field \mathbb{C} and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ be the group of residues modulo 2. An algebra \mathcal{A} is called a super-algebra if \mathcal{A} can be decomposed as the direct sum $\mathcal{A} = \mathcal{A}_{\bar{0}} \oplus \mathcal{A}_{\bar{1}}$ such that

$$\mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{0}} \subset \mathcal{A}_{\bar{0}}, \quad \mathcal{A}_{\bar{0}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{1}}, \quad \mathcal{A}_{\bar{1}} \cdot \mathcal{A}_{\bar{1}} \subset \mathcal{A}_{\bar{0}}.$$

A nonzero element of $\mathcal{A}_{\bar{0}}$ or $\mathcal{A}_{\bar{1}}$ is called *homogeneous* (respectively, even or odd). Let $p(a) = k$ if $a \in \mathcal{A}_k$ for $k \in \mathbb{Z}_2$. The number $p(a)$ is the parity of a .

The super-algebra \mathfrak{g} is a *Lie super-algebra* if it is endowed with the multiplication $[\cdot, \cdot]$ that satisfies the equalities

$$[x, y] = -(-1)^{p(x)p(y)}[y, x], \quad (20)$$

$$[x, [y, z]] = [[x, y], z] + (-1)^{p(x)p(y)}[y, [x, z]]. \quad (21)$$

here x, y , and z are arbitrary elements of \mathcal{A} and x, y are presumed homogeneous.

The super-matrix structure of a matrix is achieved whenever the parity is assigned to its rows and columns. We choose the super-matrix structure such that rows (respectively, columns) which are assigned the even parity always precede the rows (columns) of odd parity [14]. If a matrix has r even and s odd rows and p even and q odd columns, then its dimension is said to be equal to $(r \mid s) \times (p \mid q)$. In particular, we shall use the shorthand notation $(p \mid q)$ for the dimension $(p \mid q) \times (p \mid q)$. We denote by $\text{Mat}(p \mid q; \mathcal{A})$ the set of all matrices of dimension $(p \mid q)$ with elements that belong the super-algebra \mathcal{A} .

Let us introduce the super-matrix structure on the space $\text{Mat}(p \mid q; \mathcal{A})$. Consider a matrix $X = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \in \text{Mat}(p \mid q; \mathcal{A})$ and set

$$\begin{aligned} p(X) = \bar{0} & \quad \text{if } p(R_{ij}) = p(U_{ij}) = \bar{0}, \quad p(T_{ij}) = p(S_{ij}) = \bar{1}; \\ p(X) = \bar{1} & \quad \text{if } p(R_{ij}) = p(U_{ij}) = \bar{1}, \quad p(T_{ij}) = p(S_{ij}) = \bar{0}. \end{aligned}$$

Taking into account the graded skew-symmetry (20) of the bracket $[\cdot, \cdot]$, we define the Lie super-algebra structure on the space $\text{Mat}(p \mid q; \mathcal{A})$ by the formula

$$[X, Y] = XY - (-1)^{p(X)p(Y)}YX, \quad X, Y \in \text{Mat}(p \mid q; \mathcal{A}). \quad (22)$$

The Lie super-algebras $\mathfrak{gl}(m \mid n) = \text{Mat}(m \mid n, \mathbb{C})$ and $\mathfrak{sl}(m \mid n) = \{X \in \mathfrak{gl}(m \mid n) \mid \text{str } X = 0\}$, where $\text{str} \begin{pmatrix} R & S \\ T & U \end{pmatrix} = \text{tr } R - \text{tr } U$, are called the general linear and special linear Lie super-algebras, respectively.

To calculate the super-commutator $[X, Y]$ of two nonhomogeneous elements X and Y , we first split $X = X_{\bar{0}} + X_{\bar{1}}$ and $Y = Y_{\bar{0}} + Y_{\bar{1}}$ so that $p(X_{\bar{0}}) = p(Y_{\bar{0}}) = \bar{0}$ and $p(X_{\bar{1}}) = p(Y_{\bar{1}}) = \bar{1}$. Using (22), we obtain

$$\begin{aligned} [X, Y] &= [X_{\bar{0}} + X_{\bar{1}}, Y_{\bar{0}} + Y_{\bar{1}}] = [X_{\bar{0}}, Y_{\bar{0}}] + [X_{\bar{0}}, Y_{\bar{1}}] + [X_{\bar{1}}, Y_{\bar{0}}] + [X_{\bar{1}}, Y_{\bar{1}}] = \\ &= (X_{\bar{0}}Y_{\bar{0}} - Y_{\bar{0}}X_{\bar{0}}) + (X_{\bar{0}}Y_{\bar{1}} - Y_{\bar{1}}X_{\bar{0}}) + (X_{\bar{1}}Y_{\bar{0}} - Y_{\bar{0}}X_{\bar{1}}) + (X_{\bar{1}}Y_{\bar{1}} - Y_{\bar{1}}X_{\bar{1}}). \end{aligned} \quad (23)$$

The super-determinant, or the *Berezinian* of an invertible matrix $X = \begin{pmatrix} R & S \\ T & U \end{pmatrix} \in \mathfrak{gl}(m \mid n)$ is given by the formula [16]

$$\text{sdet} \begin{pmatrix} R & S \\ T & U \end{pmatrix} = \frac{\det(R - SU^{-1}T)}{\det U}.$$

Example 7. In what follows, we shall use the Lie super-algebra $\mathfrak{sl}(1 \mid 2) \simeq \mathfrak{sl}(2 \mid 1)$, see [17]. Its representation in the space $\text{Mat}(2 \mid 1; \mathbb{C})$ is given by the eight basic vectors,

four even: E^+ , E^- , H , and Z , and four odd: F^+ , F^- , \bar{F}^+ , and \bar{F}^- , where

$$\begin{aligned} E^+ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & E^- &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & H &= \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & Z &= \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ F^+ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & F^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \bar{F}^+ &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \bar{F}^- &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The elements of the basis satisfy the following commutation relations:

$$\begin{aligned} [H, E^\pm] &= \pm E^\pm & [H, F^\pm] &= \pm \frac{1}{2} F^\pm & [H, \bar{F}^\pm] &= \pm \frac{1}{2} \bar{F}^\pm \\ [Z, H] &= [Z, E^\pm] = 0 & [Z, F^\pm] &= \frac{1}{2} F^\pm & [Z, \bar{F}^\pm] &= -\frac{1}{2} \bar{F}^\pm \\ [E^\pm, F^\pm] &= [E^\pm, \bar{F}^\pm] = 0 & [E^\pm, F^\mp] &= -F^\pm & [E^\pm, \bar{F}^\mp] &= \bar{F}^\pm \\ [F^\pm, F^\pm] &= [\bar{F}^\pm, \bar{F}^\pm] = 0 & [F^\pm, F^\mp] &= [\bar{F}^\pm, \bar{F}^\mp] = 0 & [F^\pm, \bar{F}^\pm] &= E^\pm \\ [E^+, E^-] &= 2H & [F^\pm, \bar{F}^\mp] &= Z \mp H. \end{aligned}$$

The Lie super-algebra $\mathfrak{sl}(2 | 1)$ contains the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ as a subalgebra. The vectors E^\pm and H form the basis in $\mathfrak{sl}(2, \mathbb{C})$.

The Lie super-group $SL(2 | 1)$, which corresponds to the Lie super-algebra $\mathfrak{sl}(2 | 1)$, consist of the matrices with unit Berezinian: $SL(2 | 1) = \{S \in GL(2 | 1) \mid \text{sdet } S = 1\}$.

Remark 2. Consider the following three subgroups of the Lie super-group $SL(2 | 1)$:

$$G_+ = \left\{ \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \right\}, \quad G_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\}, \quad G_- = \left\{ \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \right\}.$$

Each matrix $S \in SL(2 | 1)$ can be represented [18] as a product $S = S_+ S_0 S_-$, where $S_+ \in G_+$, $S_0 \in G_0$, $S_- \in G_-$. Due to the multiplicativity of the Berezinian, $\text{sdet } S = \text{sdet } S_+ \cdot \text{sdet } S_0 \cdot \text{sdet } S_- = 1$, and in view of the obvious property $\text{sdet } S_+ = \text{sdet } S_- = 1$ for all elements of the groups G_+ and G_- , we conclude that $\text{sdet } S_0 = 1$ for all $S_0 \in G_0$.

For the Lie super-group $SL(2 | 1)$, the dimension of the matrix D is equal to 1×1 and the dimension of the matrix A is equal to 2×2 . Let us show that $G_0 \simeq SL(2 | 0)$. The condition $\text{sdet } S_0 = 1$ for the matrix $S_0 \in SL(2 | 1)$ implies the equality $\det A = \det D$ of the usual determinants of A and D . Therefore, to each matrix $A \in GL(2 | 0)$ we can put into correspondence the matrix $S_A \in G_0$ by setting $S_A = \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix}$ and conversely, to each matrix $S = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in SL(2 | 1)$ we can associate the matrix A from $GL(2 | 0)$.

2.2. Zero-curvature representations of graded extension of the KdV equation. The graded extension of the Maurer–Cartan equation 2 has the form

$$\frac{d}{dx^j} A_i - \frac{d}{dx^i} A_j + [A_i, A_j] = 0, \quad \forall i, j = 1, \dots, m : i \neq j. \quad (24)$$

Let us study in more detail the geometry of the $N=1$ and $N=2$ supersymmetry-invariant generalizations of the Korteweg–de Vries equation [2, 19].

2.2.1. $N = 1$ supersymmetric Korteweg–de Vries equation. The $N = 1$ supersymmetric generalization of the KdV equation (3) is the sKdV equation [19]

$$\phi_t = -\phi_{xxx} - 3(\phi \mathcal{D} \phi)_x, \quad \mathcal{D} = \frac{\partial}{\partial \theta} + \theta \frac{d}{dx}, \quad (25)$$

where $\phi(x, t, \theta) = \xi + \theta u$ is a complex fermionic super-field, θ is the Grassmann (or anti-commuting) variable such that $\theta^2 = 0$, the unknown u is the bosonic field, and ξ is the fermionic field. By using the expansion $\phi(x, t, \theta) = \xi + \theta u$ in (25), we obtain

$$\underline{u_t} = -u_{xxx} - 6uu_x + 3\xi\xi_{xx}, \quad (26a)$$

$$\xi_t = -\xi_{xxx} - 3(u\xi)_x. \quad (26b)$$

The KdV equation (3) is underlined in (26a).

Example 8 (Zero-curvature representation and Gardner's deformation of the sKdV equation). The sKdV equation (26) admits the $\mathfrak{sl}(2 | 1)$ -valued zero-curvature representation

$$\alpha^{N=1} = A_1^{N=1} dx + B_1^{N=1} dt,$$

where

$$A_1^{N=1} = \begin{pmatrix} -\frac{1}{2\varepsilon} & -u + \frac{1}{4\varepsilon^2} & \xi \\ 1 & -\frac{1}{2\varepsilon} & 0 \\ 0 & -\xi & -\frac{1}{\varepsilon} \end{pmatrix},$$

$$B_1^{N=1} = \begin{pmatrix} \frac{1}{2}\varepsilon^{-3} - u_x & 2u^2 + u_{xx} - \xi\xi_x + \frac{1}{2}\varepsilon^{-2}u - \frac{1}{4}\varepsilon^{-4} & -\xi_{xx} - 2\xi u - \frac{1}{2}\varepsilon^{-1}\xi_x - \frac{1}{2}\varepsilon^{-2}\xi \\ -2u - \varepsilon^{-2} & \frac{1}{2}\varepsilon^{-3} + u_x & -\xi_x - \xi\varepsilon^{-1} \\ -\xi_x + \xi\varepsilon^{-1} & \xi_{xx} + 2\xi u - \frac{1}{2}\varepsilon^{-1}\xi_x + \frac{1}{2}\varepsilon^{-2}\xi & \varepsilon^{-3} \end{pmatrix}.$$

Let us construct the generalization $S^{N=1} \in SL(2 | 1)$ of gauge transformation (19) where we had $S \in SL_2(\mathbb{C}) \simeq SL(2 | 0)$. Taking into account Remark 2, we consider the ansatz $S^{N=1} = S_+^{N=1} S_0^{N=1} S_-^{N=1}$, where $S_\alpha \in G_\alpha$, $\alpha \in \{+, 0, -\}$. Bearing in mind that $SL_2(\mathbb{C}) \simeq SL(2 | 0) \subset GL(2 | 0)$, we construct S by using following scheme:

- (1) we obtain an element $S_0^{N=1}$ by the multiplication of S from right and left by some matrices from $GL(2|0)$;
- (2) we specify the matrices $S_+^{N=1}$ and $S_-^{N=1}$.

We construct the matrix $S^{N=1}$ as follows

$$S^{N=1} = \begin{pmatrix} -1 & -\frac{1}{2}\varepsilon^{-1} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} =$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_+^{N=1}} \underbrace{\begin{pmatrix} i\sqrt{\varepsilon} & i\sqrt{\varepsilon}/\varepsilon^2 & 0 \\ 0 & i\sqrt{\varepsilon} & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix} \overbrace{\begin{pmatrix} i/\sqrt{\varepsilon} & i/(2\varepsilon\sqrt{\varepsilon}) & 0 \\ 0 & -i\sqrt{\varepsilon} & 0 \\ 1 & 0 & 1 \end{pmatrix}}^S \begin{pmatrix} 1 & \varepsilon^{-1} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_0^{N=1}} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{S_-^{N=1}}. \quad (27)$$

By applying the gauge transformation $S^{N=1}$ to the zero-curvature representation $\alpha^{N=1}$, we obtain the gauge-equivalent zero-curvature representation β for the sKdV equation (26):

$$\beta^{N=1} = (\alpha^{N=1})^{S^{N=1}} = A_2^{N=1} dx + B_2^{N=1} dt, \quad (28)$$

where

$$A_2^{N=1} = \begin{pmatrix} 0 & \varepsilon^{-1}u & \varepsilon^{-1}\xi \\ -\varepsilon & -\varepsilon^{-1} & 0 \\ 0 & \xi & -\varepsilon^{-1} \end{pmatrix},$$

$$B_2^{N=1} = \begin{pmatrix} u_x - u\varepsilon^{-1} & \frac{1}{\varepsilon}(-2u^2 - u_{xx} + \xi\xi_x) - \frac{1}{\varepsilon^2}u_x - \frac{1}{\varepsilon^3}u & \frac{1}{\varepsilon}(-\xi_{xx} - 2\xi u) - \frac{1}{\varepsilon^2}\xi_x - \frac{1}{\varepsilon^3}\xi \\ 2u\varepsilon + \varepsilon^{-1} & u_x + u\varepsilon^{-1} + \varepsilon^{-3} & \xi_x + \xi\varepsilon^{-1} \\ -\xi_{;x}\varepsilon + \xi & -\xi_{xx} - 2\xi u & \varepsilon^{-3} \end{pmatrix}.$$

Let us recall that formula (9) yields the representation of the matrices $A_2^{N=1}$ and $B_2^{N=1}$ in terms of vector fields. By this argument, from the zero-curvature representation $\beta^{N=1}$ we obtain the two-dimensional covering over the sKdV equation (26); one of the two new nonlocal variables is bosonic (let us denote it by \tilde{u}) and the other, $\tilde{\xi}$ is fermionic:

$$\begin{aligned} \tilde{u}_x &= -\tilde{u}^2\varepsilon + (\tilde{u} - u)\varepsilon^{-1} - \tilde{\xi}\xi, \\ \tilde{\xi}_x &= -\tilde{\xi}\tilde{u}\varepsilon + (\tilde{\xi} - \xi)\varepsilon^{-1}, \\ \tilde{u}_t &= \frac{1}{\varepsilon^3}(2\tilde{u}^2u\varepsilon^4 + \tilde{u}^2\varepsilon^2 - 2\tilde{u}u\varepsilon^2 - 2\tilde{u}u_x\varepsilon^3 - \tilde{u} + 2u^2\varepsilon^2 + u + u_{xx}\varepsilon^2 + u_x\varepsilon - \tilde{\xi}\tilde{u}\xi_x\varepsilon^4 + \tilde{\xi}\xi_{xx}\varepsilon^3 + \\ &\quad + \tilde{\xi}\xi\tilde{u}\varepsilon^3 + 2\tilde{\xi}\xi u\varepsilon^3 - \xi\xi_x\varepsilon^2), \\ \tilde{\xi}_t &= \frac{1}{\varepsilon^3}(-\tilde{u}\xi_x\varepsilon^3 + \xi_{xx}\varepsilon^2 + \xi_x\varepsilon + 2\tilde{\xi}\tilde{u}u\varepsilon^4 + \tilde{\xi}\tilde{u}\varepsilon^2 - \xi u\varepsilon^2 - \xi u_x\varepsilon^3 - \tilde{\xi} - \xi\tilde{u}\varepsilon^2 + 2\xi u\varepsilon^2 + \xi). \end{aligned}$$

We now express the local variables u and ξ from \tilde{u}_x and $\tilde{\xi}_x$ and substitute them in \tilde{u}_t and $\tilde{\xi}_t$. We thus obtain the Gardner deformation [3] of the sKdV equation (26):

$$\begin{aligned} \mathcal{E}_\varepsilon &= \left\{ \tilde{u}_t = \underline{6\tilde{u}^2\tilde{u}_x\varepsilon^2 - 6\tilde{u}\tilde{u}_x - \tilde{u}_{xxx}} - 3\tilde{\xi}\tilde{u}\tilde{\xi}_{xx}\varepsilon^2 + 3\tilde{\xi}\tilde{\xi}_{xx} - 3\tilde{\xi}\xi_x\tilde{u}_x\varepsilon^2, \right. \\ &\quad \left. \tilde{\xi}_t = 3\tilde{u}^2\tilde{\xi}_x\varepsilon^2 - 3\tilde{u}\tilde{\xi}_x - \tilde{\xi}_{xxx} + 3\tilde{\xi}\tilde{u}\tilde{u}_x\varepsilon^2 - 3\tilde{\xi}\tilde{u}_x \right\}, \\ \mathfrak{m}_\varepsilon &= \left\{ u = \underline{\tilde{u} - \varepsilon\tilde{u}_x} + \varepsilon^2(\tilde{\xi}\tilde{\xi}_x\varepsilon^2 - \tilde{u}^2), \quad \xi = \tilde{\xi} - \varepsilon\tilde{\xi}_x - \varepsilon^2\tilde{\xi}\tilde{u} \right\}: \mathcal{E}_\varepsilon \rightarrow \mathcal{E}_{\text{sKdV}}. \end{aligned}$$

The original Gardner deformation (11) of the KdV equation (3) is underlined in the above formulas.

In Example 8 we generalized, to the graded case, the scheme of constructing Gardner's deformations from known zero-curvature representations (see Example 6).

2.2.2. $N = 2$ supersymmetric Korteweg–de Vries equation. Let us consider the four-component generalization of the KdV equation (3), namely, the $N=2$ supersymmetric Korteweg–de Vries equation (SKdV) [2]:

$$\mathbf{u}_t = -\mathbf{u}_{xxx} + 3(\mathbf{u}\mathcal{D}_1\mathcal{D}_2\mathbf{u})_x + \frac{a-1}{2}(\mathcal{D}_1\mathcal{D}_2\mathbf{u}^2)_x + 3a\mathbf{u}^2\mathbf{u}_x, \quad \mathcal{D}_i = \frac{\partial}{\partial\theta_i} + \theta_i \cdot \frac{d}{dx}, \quad (29)$$

where

$$\mathbf{u}(x, t; \theta_1, \theta_2) = u_0(x, t) + \theta_1 \cdot u_1(x, t) + \theta_2 \cdot u_2(x, t) + \theta_1\theta_2 \cdot u_{12}(x, t) \quad (30)$$

is the complex bosonic super-field, θ_1, θ_2 are Grassmann variables such that $\theta_1^2 = \theta_2^2 = \theta_1\theta_2 + \theta_2\theta_1 = 0$, u_0, u_{12} are bosonic fields, and u_1, u_2 are fermionic fields. Expansion (30)

converts (29) to the four-component system

$$u_{0;t} = -u_{0;xxx} + (au_0^3 - (a+2)u_0u_{12} + (a-1)u_1u_2)_x, \quad (31a)$$

$$u_{1;t} = -u_{1;xxx} + ((a+2)u_0u_{2;x} + (a-1)u_{0;x}u_2 - 3u_1u_{12} + 3au_0^2u_1)_x, \quad (31b)$$

$$u_{2;t} = -u_{2;xxx} + (-(a+2)u_0u_{1;x} - (a-1)u_{0;x}u_1 - 3u_2u_{12} + 3au_0^2u_2)_x, \quad (31c)$$

$$\underline{u_{12;t}} = -\underline{u_{12;xxx}} - 6u_{12}u_{12;x} + 3au_{0;x}u_{0;xx} + (a+2)u_0u_{0;xxx} + 3u_1u_{1;xx} + 3u_2u_{2;xx} + 3a(u_0^2u_{12} - 2u_0u_1u_2)_x. \quad (31d)$$

The KdV equation is underlined in (31d). The SKdV equation is most interesting (in particular, bi-Hamiltonian, whence completely integrable) if $a \in \{-2, 1, 4\}$, see [2, 4, 20]. Let us consider the bosonic limit $u_1 = u_2 = 0$ of system (31): by setting $a = -2$ we obtain the triangular system which consists of the modified KdV equation upon u_0 and the equation of KdV-type; in the case $a = 1$ we obtain the Krasil'shchik–Kersten system; for $a = 4$, we obtain the third equation in the Kaup–Boussinesq hierarchy. A Gardner deformation of the Kaup–Boussinesq system was constructed in [4].

The Gardner deformation problem for the $N = 2$ supersymmetric $a = 4$ KdV equation was formulated by P. Mathieu in [2]. In the paper [4] it was shown that one can not construct such a deformation under the assumptions that, first, the deformation is polynomial in \mathcal{E} , second, it involves only the super-fields but not their components, and third, it contains the known deformation (11) under the reduction $u_0 = 0$, $u_1 = u_2 = 0$. Therefore, we shall find a graded generalization of Gardner's deformation (11) for the system of four equations (31) treating it in components but not as the single equation (29) upon the super-field.

The SKdV equation (31) admits [21] the $\mathfrak{sl}(1 | 2)$ -valued zero-curvature representation $\alpha^{N=2} = Adx + Bdt$ such that

$$A = \begin{pmatrix} \eta - iu_0 & \eta^2 - 2i\eta u_0 - u_0^2 - u_{12} & -u_2 - iu_1 \\ 1 & \eta - iu_0 & 0 \\ 0 & u_2 - u_1i & 2\eta - 2iu_0 \end{pmatrix}, \quad (32a)$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (32b)$$

where the elements of B are as follows:

$$\begin{aligned}
b_{11} &= -4\eta^3 - 2\eta u_{0;x}i - 4u_0^3i + 6u_0u_{12}i + 4u_0u_{0;x} + u_{0;xx}i - u_{12;x} + 4u_2u_1i, \\
b_{12} &= -4\eta^4 + 4\eta^3u_0i + 2\eta^2u_{12} - 4\eta u_0^3i + 8\eta u_0u_{12}i + 2\eta u_{0;xx}i - 4u_0^4 - 2u_0^2u_{12} - \\
&\quad - 4u_0u_{0;xx} + 2u_{12}^2 - 4u_{0;x}^2 + u_{12;xx} - u_2u_{2;x} + 4u_2u_1\eta i - 8u_2u_1u_0 - u_1u_{1;x}, \\
b_{13} &= -\eta u_{2;x} - \eta u_{1;x}i - 5u_0u_{2;x}i + 5u_0u_{1;x} + u_{2;xx} + u_{1;xx}i + 2u_2\eta^2 + 2u_2\eta u_0i - \\
&\quad - 8u_2u_0^2 + 2u_2u_{12} - 4u_2u_{0;x}i + 2u_1\eta^2i - 2u_1\eta u_0 - 8u_1u_0^2i + 2u_1u_{12}i + 4u_1u_{0;x}, \\
b_{21} &= 2(-2\eta^2 - 2\eta u_0i + 2u_0^2 - u_{12}), \\
b_{22} &= -4\eta^3 + 2\eta u_{0;x}i - 4u_0^3i + 6u_0u_{12}i - 4u_0u_{0;x} + u_{0;xx}i + u_{12;x} + 4u_2u_1i, \\
b_{23} &= u_{2;x} + u_{1;x}i - 2u_2\eta - 4u_2u_0i - 2u_1\eta i + 4u_1u_0, \\
b_{31} &= u_{2;x} - u_{1;x}i + 2u_2\eta + 4u_2u_0i - 2u_1\eta i + 4u_1u_0, \\
b_{32} &= -\eta u_{2;x} + \eta u_{1;x}i - 5u_0u_{2;x}i - 5u_0u_{1;x} - u_{2;xx} + u_{1;xx}i - 2u_2\eta^2 - 2u_2\eta u_0i + \\
&\quad + 8u_2u_0^2 - 2u_2u_{12} - 4u_2u_{0;x}i + 2u_1\eta^2i - 2u_1\eta u_0 - 8u_1u_0^2i + 2u_1u_{12}i - 4u_1u_{0;x}, \\
b_{33} &= 2(-4\eta^3 - 4u_0^3i + 6u_0u_{12}i + u_{0;xx}i + 4u_2u_1i).
\end{aligned}$$

Remark 3. Let us recall that the vectors Z , H and E^\pm that belongs to $\mathfrak{sl}(2 | 1)$,

$$Z = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

generate a basis in $\mathfrak{gl}(2, \mathbb{C})$. The vector Z commutes with any other vector from $\mathfrak{gl}(2, \mathbb{C})$.

The reduction $u_0 = u_1 = u_2 = 0$ converts zero-curvature representation (32) to the $\mathfrak{gl}(2, \mathbb{C})$ -valued zero-curvature representation of the KdV equation (3),

$$\begin{aligned}
A_{\text{KdV}} &= \begin{pmatrix} \eta & \eta^2 - u_{12} & 0 \\ 1 & \eta & 0 \\ 0 & 0 & 2\eta \end{pmatrix}, \\
B_{\text{KdV}} &= \begin{pmatrix} -4\eta^3 - u_{12;x} & -4\eta^4 + 2\eta^2u_{12} + 2u_{12}^2 + u_{12;xx} & 0 \\ 2(-2\eta^2 - u_{12}) & -4\eta^3 + u_{12;x} & 0 \\ 0 & 0 & -8\eta^3 \end{pmatrix}.
\end{aligned}$$

Taking into account Remark 3, we obtain the $\mathfrak{sl}(2, \mathbb{C})$ -valued zero-curvature representation (14) for the KdV equation (3) by omitting the summands $\eta \otimes Z dx$ and $-4\eta^3 \otimes Z dt$ in A_{KdV} and B_{KdV} and by denoting $\eta^2 = \lambda$.

Proposition 1. *The $N=2$ supersymmetric $a=4$ Korteweg-de Vries equation (31) admits the covering that, under the reduction $u_0 = u_1 = u_2 = 0$ of (31) to the KdV equation (3), reduces to the covering that contains, in the form of (12), the known Gardner deformation of the KdV equation (3).*

Proof. Let us extend the gauge transformation (19), which was determined by the element S of the Lie group $SL(2, \mathbb{C})$. We let

$$S^{N=2} = \begin{pmatrix} -1 & -\frac{1}{2}\varepsilon^{-1} & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & -\varepsilon \end{pmatrix}. \quad (33)$$

Acting by gauge transformation (33) on zero-curvature representation (32), we obtain the graded zero-curvature representation that contains the “small” zero-curvature representation which originates from (12) and gauge-equivalent to (14) for the KdV equation (3). Specifically, we have that

$$A = \begin{pmatrix} iu_0 & \varepsilon^{-1}(u_0^2 + u_{12}) - i\varepsilon^{-2}u_0 & \varepsilon^{-1}(u_2 - iu_1) \\ -\varepsilon & iu_0 - \varepsilon^{-1} & 0 \\ 0 & u_2 + iu_1 & 2iu_0 - \varepsilon^{-1} \end{pmatrix}, \quad (34a)$$

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \quad (34b)$$

where the elements of the matrix B are as follows

$$\begin{aligned} b_{11} &= 4iu_0^3 - 6iu_0u_{12} + 4u_0u_{0;x} - iu_{0;xx} - u_{12;x} - 4iu_2u_1 + \varepsilon^{-1}(2u_0^2 - u_{12} - iu_{0;x}) - i\varepsilon^{-2}u_0, \\ b_{12} &= \varepsilon^{-1}(4u_0^4 + 2u_0^2u_{12} + 4u_0u_{0;xx} - 2u_{12}^2 + 4u_{0;x}^2 - u_{12;xx} + u_2u_{2;x} + 8u_2u_1u_0 + u_1u_{1;x}) + \\ &\quad + \varepsilon^{-2}(2iu_0^3 - 4iu_0u_{12} + 4u_0u_{0;x} - iu_{0;xx} - u_{12;x} - 2iu_2u_1) + \varepsilon^{-3}(u_0^2 - u_{12} - iu_{0;x}) - \\ &\quad - i\varepsilon^{-4}u_0, \\ b_{13} &= \varepsilon^{-1}(-5iu_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} + iu_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} - 4iu_2u_{0;x} - 8iu_1u_0^2 + \\ &\quad + 2iu_1u_{12} - 4u_1u_{0;x}) + \varepsilon^{-2}(-u_{2;x} + iu_{1;x} - 3iu_2u_0 - 3u_1u_0) + \varepsilon^{-3}(-u_2 + iu_1), \\ b_{21} &= 2\varepsilon(-2u_0^2 + u_{12}) + 2iu_0 + \varepsilon^{-1}, \\ b_{22} &= 4iu_0^3 - 6iu_0u_{12} - 4u_0u_{0;x} - iu_{0;xx} + u_{12;x} - 4iu_2u_1 + \varepsilon^{-1}(-2u_0^2 + u_{12} + iu_{0;x}) + \\ &\quad + i\varepsilon^{-1}u_0 + \varepsilon^{-3}, \\ b_{23} &= u_{2;x} - iu_{1;x} + 4iu_2u_0 + 4u_1u_0 + \varepsilon^{-1}(u_2 - iu_1), \\ b_{31} &= \varepsilon(-u_{2;x} - iu_{1;x} + 4iu_2u_0 - 4u_1u_0) + u_2 + iu_1, \\ b_{32} &= 5iu_0u_{2;x} - 5u_0u_{1;x} - u_{2;xx} - iu_{1;xx} + 8u_2u_0^2 - 2u_2u_{12} + 4iu_2u_{0;x} + 8iu_1u_0^2 - 2iu_1u_{12} - \\ &\quad - 4u_1u_{0;x} + \varepsilon^{-1}u_0(iu_2 - u_1), \\ b_{33} &= 2(4iu_0^3 - 6iu_0u_{12} - iu_{0;xx} - 4iu_2u_1) + \varepsilon^{-3}. \end{aligned}$$

The projective substitution (8) yields the two-dimensional covering over the $N=2$, $a=4$ SKdV equation. Under the reduction $u_0 = u_1 = u_2 = 0$, the covering contains (12), which is equivalent to Gardner's deformation (11) of the KdV equation (3). The x -components of the derivation rules for the nonlocalities w and f are

$$\begin{aligned} w_x &= \underline{-\varepsilon w^2 + \varepsilon^{-1}(w - u_{12})} - fu_2 - ifu_1 - \varepsilon^{-1}u_0^2 - \varepsilon^{-2}iu_0, \\ f_x &= -\varepsilon wf - iu_0f + \varepsilon^{-1}(f - u_2 + iu_1), \end{aligned}$$

here and in what follows we underline the covering (12) that encodes the “small” Gardner deformation for the KdV equation. The t -components of the “large” covering over the $N=2$, $a=4$ SKdV are

$$\begin{aligned}
w_t = & \varepsilon(-4w^2u_0^2 + \underline{2w^2u_{12}} - fwu_{2;x} - ifwu_{1;x} + 4ifu_2wu_0 - 4fu_1wu_0) + 2iw^2u_0 + \\
& + 8wu_0u_{0;x} - \underline{2wu_{12;x}} - 5ifu_0u_{2;x} + 5fu_0u_{1;x} + fu_{2;xx} + ifu_{1;xx} + fu_2w - 8fu_2u_0^2 + \\
& + 2fu_2u_{12} - 4ifu_2u_{0;x} + ifu_1w - 8ifu_1u_0^2 + 2ifu_1u_{12} + 4fu_1u_{0;x} + \varepsilon^{-1}(\underline{w^2} + 4wu_0^2 - \\
& - \underline{2wu_{12}} - 2iwu_{0;x} - 4u_0^4 - 2u_0^2u_{12} - 4u_0u_{0;xx} + \underline{2u_{12}^2} - 4u_{0;x}^2 + \underline{u_{12;xx}} - ifu_2u_0 + \\
& + fu_1u_0 - u_2u_{2;x} - 8u_2u_1u_0 - u_1u_{1;x}) + \varepsilon^{-2}(-2iwu_0 - 2iu_0^3 + 4iu_0u_{12} - 4u_0u_{0;x} + \\
& + iu_{0;xx} + \underline{u_{12;x}} + 2iu_2u_1) + \varepsilon^{-3}(\underline{-w} - \underline{u_0^2 + u_{12}} + iu_{0;x}) + \varepsilon^{-4}iu_0, \\
f_t = & 2\varepsilon w(-2fu_0^2 + fu_{12}) + (-wu_{2;x} + iwu_{1;x} + 2ifwu_0 - 4ifu_0^3 + 6ifu_0u_{12} + 4fu_0u_{0;x} + \\
& + ifu_{0;xx} - fu_{12;x} + 4ifu_2u_1 - 4iu_2wu_0 - 4u_1wu_0) + \varepsilon^{-1}(5iu_0u_{2;x} + 5u_0u_{1;x} + u_{2;xx} - \\
& - iu_{1;xx} + fw + 2fu_0^2 - fu_{12} - ifu_{0;x} - u_2w - 8u_2u_0^2 + 2u_2u_{12} + 4iu_2u_{0;x} + iu_1w + \\
& + 8iu_1u_0^2 - 2iu_1u_{12} + 4u_1u_{0;x}) + \varepsilon^{-2}(u_{2;x} - iu_{1;x} - ifu_0 + 3iu_2u_0 + 3u_1u_0) + \\
& + \varepsilon^{-3}(-f + u_2 - iu_1).
\end{aligned}$$

This proves our claim. \square

We finally remark that the reduction $u_0 = 0$, $u_1 = 0$ (and the change of notation $u_2 \rightarrow \xi$, $u_{12} \rightarrow u$) maps this covering over the $N=2$, $a=4$ SKdV equation to the covering which was constructed in Example 8 for the $N=1$ supersymmetric Korteweg–de Vries equation (25).

CONCLUSION

By now the Gardner deformation problem for the $N=2$ supersymmetric $a=4$ Korteweg–de Vries equation ([3]) is solved. In this paper we have found the solution which is an alternative to our previous result in [4]. Namely, we introduced the nonlocal bosonic and fermionic variables in such a way that the rules to differentiate them are consistent by virtue of the super-equation at hand and second, the entire system retracts to the standard KdV equation and the classical Gardner deformation for it ([1]) under setting to zero the fermionic nonlocal variable and the first three components of the $N=2$ superfield in (29). At the same time, the nonlocal structure under study is equivalent to the $\mathfrak{sl}(2 | 1)$ -valued zero-curvature representation for this super-equation; the zero-curvature representation contains the non-removable spectral parameter, which manifest the integrability.

Our new solution of P. Mathieu’s Open problem 2 (see [3]) relies on the interpretation of both Gardner’s deformations and zero-curvature representations in similar terms, as a specific type of nonlocal structures over the equation of motion. However, we emphasize that generally there is no one-to-one correspondence between the two constructions, so that the interpretation of deformations in the Lie-algebraic language is not always possible. Because this correlation between the two approaches to the integrability was not revealed in the canonical formulation of the deformation problem [3], there appeared some attempts to solve it within the classical scheme but the progress was partial. Still,

the use of zero-curvature representations in this context could have given the sought deformation long ago.

Let us also notice that projective substitution (8) correlates the super-dimension of the Lie algebra in a zero-curvature representation for a differential equation with the numbers of bosonic and fermionic nonlocalities over the same system: a subalgebra of $\mathfrak{gl}(p | q)$ yields at most $p - 1$ bosonic and q fermionic variables. This implies that, for a covering over the $N=2$ supersymmetric KdV equation (29) to extend the Gardner deformation (11) in its classical sense $\mathfrak{m}_\varepsilon : \mathcal{E}_\varepsilon \rightarrow \mathcal{E}$ (see [1, 2, 7]), the extension \mathcal{E}_ε must be the system of evolution equations upon two bosonic and two fermionic fields. Therefore, one may have to use the $\mathfrak{sl}(3 | 2)$ -valued zero-curvature representations. This outlines the working approach to a yet another method of solving the Gardner deformation problem for the $N=2$ supersymmetric Korteweg–de Vries systems (29), which we leave as a new open problem.

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